Quantum Information Methods for Many-Body Physics

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Exercise Sheet 5: Haar Averages and 2k-OTOCs **Due: 7 July, 10:00**

Out-of-time-ordered correlators (OTOCs) are widely used to characterize information scrambling and quantum chaos. In this sheet, we study the Haar-averaged 2k-OTOCs

$$\mathcal{O}_{2k} = \mathbb{E}_{U\in\mathcal{U}(d)} \frac{1}{d} \mathrm{tr}\left(A_U B A_U B \cdots A_U B\right)$$

where $A_U = U^\dagger A U$, and $A, B \in \mathcal{B}(\mathbb{C}^d)$ are fixed operators. We work at infinite temperature, and express the result in terms of normalized traces and free cumulants.

1 OTOC as a trace with the Haar channel (2 P)

Show that the average

$$\mathcal{O}_{2k} = \mathbb{E}_U \frac{1}{d} \mathrm{tr}(A_U B A_U B \cdots A_U B)$$

can be rewritten as

$$\mathcal{O}_{2k} = \mathrm{tr}\left[\mathcal{A}\, \mathcal{M}_k(\mathcal{B})\right]$$

for certain $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{M}_k$ is the Haar k-fold channel:

$$\mathcal{M}_k(\mathcal{B}) = \int dU\, (U^\dagger)^{\otimes k} B^{\otimes k} U^{\otimes k} .$$

Who are the operators $\mathcal{A}$ and $\mathcal{B}$? *Hint*: Use the cyclicity of the trace and the replica trick, e.g. $\mathrm{Tr}(A^3) = \mathrm{Tr}(A^{\otimes 3} R_{(123)})$.

2 Compute 2k-OTOC for $A = B$ (3 P)

Set $A_U = U^\dagger A U$ and consider $\mathcal{O}_{2k} = \mathbb{E}_U \frac{1}{d} \mathrm{tr}(A_U A A_U A \cdots A_U A)$.
Compute explicitly $\mathcal{O}_{2k}$ for $k = 1, 2, 3$.

3 Leading order in $1/d$ (2 P)

From the results above, assume that $\mathrm{tr}(A^k) = d a_k$. Identify the leading order in $1/d$ for the $2k$-OTOC for $k = 1, 2, 3$.

4 Generic operators $A_U = U^\dagger A U$ and B (3 P)

Now consider generic A and B, with $\mathrm{tr}(A) = \mathrm{tr}(B) = 0$, and compute

$$\mathcal{O}_{2k} = \mathbb{E}_U \frac{1}{d} \mathrm{tr}\left(A_U B A_U B \cdots A_U B\right)$$

for $k = 1, 2, 3$. Again, assuming $\mathrm{Tr}(A^k) = d a_k$ and $\mathrm{Tr}(B^k) = d b_k$ (note, $a_1 = 0 = b_1$ in this case), compute the leading order in $1/d$.