Quantum Information Methods for Many-Body Physics

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CHAPTER 1

INTRODUCTION

Quantum information science is the field that studies how information is stored, processed, and transmitted when it is governed by the laws of quantum mechanics. It includes areas such as *quantum computing*, *quantum communication, quantum cryptography, quantum sensing, and quantum error correction*. Although still relatively young, quantum information science has already had a profound impact on many areas of physics, especially on the study of quantum systems composed of many interacting particles, commonly referred to as *many-body quantum systems*.

Within this context, *quantum circuits* and *tensor networks* have emerged as essential tools to tackle fundamental questions in quantum dynamics—from the mechanisms of *thermalization* and the emergence of *statistical mechanics* in isolated systems, to the onset of *quantum chaos* and its deep connections with *black hole physics and holography* in quantum gravity. Studying these phenomena in a concrete setting is notoriously difficult, due to the exponential growth of the Hilbert space and the inherently non-linear structure of quantum correlations. *Random quantum circuits* and *tensor networks* offer a powerful way to overcome these challenges: they enable analytical and numerical progress through disorder averaging, while capturing the typical behavior of generic quantum systems thanks to *quantum typicality arguments*.

Crucially, the interplay between quantum information and many-body physics has not only refined our understanding of traditional problems, but has also uncovered entirely new dynamical phases of matter – phases that arise uniquely in *programmable quantum devices*. This so-called *synthetic quantum matter* cannot be characterized by conventional local order parameters such as magnetization or current. Instead, its defining features are quantum informational, such as the structure of entanglement or nonstabilizer (magic state) resources, or the system's ability to preserve quantum information against noise and local errors. Understanding and classifying such phases requires a shift in perspective –from symmetry and energetics to *information content* and *computational complexity*.

At the same time, random unitary dynamics, especially in the form of random quantum circuits, have become indispensable tools in the *NISQ (noisy intermediate-scale quantum)* era. They provide efficient and versatile frameworks for a wide range of applications, including the benchmarking and verification of quantum computations, the characterization of noise in experimental platforms, and the estimation of observables via shadow tomography. Far from being purely theoretical constructs, these methods are implemented across various platforms – from superconducting qubits to cold atoms – and are central to the ongoing development of near-term quantum technologies.

This course provides a pedagogical introduction to random unitaries and to several key methods from quantum information theory, with a focus on their application to many-body physics. A substantial part of the course covers research-level topics introduced only in the past few years, offering a unique opportunity to engage with current questions at the interface between two rapidly evolving fields. It is also intended to serve as a solid preparation to pursue a Master's thesis, a doctorate or work in the private for these areas.

1.1 Overview

The structure of the course is as follows. We begin with a chapter on permutations and a graphical calculus, which will provide the foundation for the treatment of randomization methods throughout the course. We then introduce the core randomization concepts in two parts, Quantum Randomness I and II—each followed by a chapter that connects the methods to applications in many-body systems.

1. In Quantum Randomness I (Ch. 2), we introduce *Weingarten calculus*, the central toolbox for computing averages over the unitary group, which naturally arise when considering statistical

properties of random evolutions. We also *study unitary designs*, which provide efficient approximations of Haar randomness and play an important role in practical implementations.

2. In Ch. 3, we present the *framework of classical shadows*. Introduced around 2020, classical shadows offer an efficient way to extract information about quantum states using only a small number of randomized measurements. This technique has already found widespread use in experimental platforms and continues to inspire a growing body of research.

3. In Quantum Randomness II (Ch. 4), we explore random quantum circuits in detail. These models provide an efficient and physically motivated approach to generate randomness in quantum many-body systems, and they serve as minimal models for chaotic quantum dynamics.

4. In Ch. 5, we show how random circuits can be used to model scrambling, thermalization, and information spreading in interacting quantum systems. These models also offer connections to quantum chaos, complexity growth, and typicality in many-body physics.

Each chapter concludes with a short guide to the research literature, primarily in the form of original articles, allowing students to explore further and connect the course material to current work in the field.

1.2 Permutations and their combinatorics

Why permutations? A central theme of this course is the study of random unitaries and their applications in quantum many-body systems. To understand their behavior, we need to analyze the statistics of random unitaries—specifically, we are interested in computing averages, variances, and higher moments of functions involving random unitary matrices.

At first glance, this might seem daunting: computing integrals over the unitary group is, in general, a highly nontrivial task. However, a powerful insight from representation theory, known as *Schur-Weyl duality*, provides a way forward. This duality reveals a deep connection *between the action of the unitary group and the action of the permutation group*, which allows us to reformulate complicated integrals in terms of combinatorics of permutations.

This leads to the framework known as *Weingarten calculus*, which enables the exact evaluation of many relevant averages over the unitary group. As a result, permutations will play a key role throughout this course – not for abstract mathematical reasons, but because they offer a concrete and computable handle on random quantum processes.

In this section, we will introduce the essential properties of permutations needed for our purposes. While there is a rich and beautiful mathematical structure behind these ideas, we will focus only on the aspects that are directly relevant for our discussion and applications. The interested reader can consult the bibliography.

1.2.1 Permutations and Cycles

A *permutation* is a reordering of a finite set of elements. In this course, we consider permutations of the set $\{1, 2, \ldots, k\}$. We will denote permutations by Greek letters such as π, σ, τ, and so on. The set of all permutations of k elements forms a group under composition, called the *symmetric group*, and denoted by S_k.

Given a permutation $\sigma \in S_k$ and an element $x \in \{1, \ldots, k\}$, we write $\sigma(x)$ to indicate the image of x under σ. A common and explicit way to write a permutation is the *two-line notation*, where the first row lists the original elements and the second row gives their images under the permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & k \\ \sigma(1) & \sigma(2) & \ldots & \sigma(k) \end{pmatrix}. \tag{1.1}$$

Example 1.2.1: Explicit Notation of Permutations

An example of a permutation of four elements is:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}. \tag{1.2}$$

This means that the permutation acts as:

$$\sigma(1) = 3, \quad \sigma(2) = 1, \quad \sigma(3) = 4, \quad \sigma(4) = 2. \tag{1.3}$$

The group operation in S_k is the composition of permutations, denoted by $\sigma \cdot \tau$ for $\sigma, \tau \in S_k$, and defined by

$$(\sigma \cdot \tau)(x) = \sigma(\tau(x)) \qquad \text{for all } x \in \{1, \ldots, k\}. \tag{1.4}$$

Note that permutation composition is applied from right to left: τ acts first, followed by σ.

Example 1.2.2: Product of Permutations

Consider the following two permutations of $k = 4$ elements:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}. \tag{1.5}$$

To compute the composition $(\tau \cdot \sigma)(x) = \tau(\sigma(x))$, we apply σ first and then τ:

$$\begin{aligned} \sigma(1) &= 3, \quad \tau(\sigma(1)) = \tau(3) = 3, \\ \sigma(2) &= 1, \quad \tau(\sigma(2)) = \tau(1) = 4, \\ \sigma(3) &= 4, \quad \tau(\sigma(3)) = \tau(4) = 1, \\ \sigma(4) &= 2, \quad \tau(\sigma(4)) = \tau(2) = 2. \end{aligned} \tag{1.6}$$

Putting everything together, we find:

$$\tau \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}. \tag{1.7}$$

It is important to emphasize that for $k \geq 3$, the symmetric group S_k is *non-abelian*, meaning that the order of composition matters – in general, $\sigma \cdot \tau \neq \tau \cdot \sigma$.

Exercise 1.1. *Consider the permutations from Example 1.2.2. Compute the product $\sigma \cdot \tau$, and verify that it differs from $\tau \cdot \sigma$.*

The symmetric group contains a special element called the *identity permutation*, denoted by ι, which leaves all elements unchanged:

$$\iota = \begin{pmatrix} 1 & 2 & \ldots & k \\ 1 & 2 & \ldots & k \end{pmatrix}. \tag{1.8}$$

By definition, composition with the identity does not change the permutation:

$$\iota \cdot \sigma = \sigma = \sigma \cdot \iota. \tag{1.9}$$

Moreover, every permutation $\sigma \in S_k$ has an *inverse* σ^{-1} such that:

$$\sigma \cdot \sigma^{-1} = \iota = \sigma^{-1} \cdot \sigma. \tag{1.10}$$

To compute the inverse of a permutation π, for each index $i \in \{1, 2, \ldots, k\}$ we set $\pi^{-1}(\pi(i)) = \iota(i) = i$. The resulting list π^{-1} is the inverse permutation.

Example 1.2.3: Inverse of a Permutation

Consider the following two permutations of $k = 5$ elements:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}. \tag{1.11}$$

To compute the inverse σ^{-1} we use the rule $\sigma^{-1}(\sigma(i)) \equiv i$

$$\begin{aligned} \sigma(1) = 3 &\Rightarrow \sigma^{-1}(3) = 1 \\ \sigma(2) = 5 &\Rightarrow \sigma^{-1}(5) = 2 \\ \sigma(3) = 1 &\Rightarrow \sigma^{-1}(1) = 3 \\ \sigma(4) = 2 &\Rightarrow \sigma^{-1}(2) = 4 \\ \sigma(5) = 4 &\Rightarrow \sigma^{-1}(4) = 5 \end{aligned} \tag{1.12}$$

Reordering the list in terms of the argument, we find

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}. \tag{1.13}$$

A *cyclic permutation*, or simply a *cycle*, is a specific type of permutation in which a subset of elements is permuted in a closed loop, while all remaining elements remain fixed. Formally, an l-cycle is a permutation that permutes r elements cyclically and leaves the other $k - l$ elements unchanged. The number l is called the *length of the cycle*.

Concretely, a cycle of length l means that there exists a subset $\{i_1, i_2, \ldots, i_l\} \subset \{1, 2, \ldots, k\}$ such that

$$\sigma(i_1) = i_2, \quad \sigma(i_2) = i_3, \quad \ldots, \quad \sigma(i_{l-1}) = i_l, \quad \sigma(i_l) = i_1, \tag{1.14}$$

and for all other elements $x \notin \{i_1, \ldots, i_l\}$, we have $\sigma(x) = x$.

Exercise 1.2 (Cyclic permutations)**.** *The following are examples of cyclic permutations:*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix}. \tag{1.15}$$

Can you identify the subset $\{a_1, \ldots, a_l\}$ that is cyclically permuted in each case? What is the length l of each cycle? [Answer: For τ, $r = l$; for σ, $r = l$.]

One fundamental property of cycles is that any permutation can be decomposed into a product of disjoint cycles. That is, for any $\sigma \in S_k$, there exists a unique set of cycles that act on mutually disjoint subsets of $\{1, 2, \ldots, k\}$. This decomposition is unique up to the order in which the cycles are written. This motivates the *cycle notation* of permutations:

$$\sigma = (a_1\, a_2\, \ldots\, a_{j_1})(a_{j_1+1}\, a_{j_1+2}\, \ldots\, a_{j_2}) \ldots (a_{j_{r-1}+1}\, a_{j_{r-1}+2}\, \ldots\, a_{j_r})\,. \tag{1.16}$$

Here, each tuple represents a cycle, and all elements a_m are drawn from $\{1, 2, \ldots, k\}$ without repetition. The integer $r = \#(\sigma)$ denotes the *number of disjoint cycles* in the decomposition of σ.

Let us now describe an explicit algorithm to obtain the cycle decomposition of a permutation, as in Eq. (1.16). The idea is simple: we iteratively follow the action of the permutation until we return to the starting point, keeping track of all visited elements.

(i) **Start from the smallest unvisited element.** Initially, this is $x = 1$. If 1 has already been included in a previous cycle, move to the next smallest unvisited element.

(ii) **Construct a cycle by iterating the permutation.** Begin by writing down x. Then repeatedly apply the permutation π to generate the sequence

$$x,\ \pi(x),\ \pi(\pi(x)),\ \pi(\pi(\pi(x))),\ \ldots$$

Continue this process until you return to the starting point x. The list of elements

$$(x\ \ \pi(x)\ \ \pi^2(x)\ \ \ldots\ \ \pi^{j-1}(x)) \tag{1.17}$$

forms a cycle of length j. Mark all of these elements as visited.

(iii) **Repeat the process.** Find the next smallest unvisited element and return to step (ii). Continue until all elements have been visited. The full cycle decomposition of π is then obtained by combining the individual cycles found in each iteration.

Example 1.2.4: Cycle decomposition

Consider the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}. \tag{1.18}$$

We apply the cycle decomposition algorithm step by step:

- Start with 1:
$$1 \overset{\tau}{\mapsto} 4 \overset{\tau}{\mapsto} 1.$$
This gives the first cycle: $(1\ 4)$. Mark 1 and 4 as visited.
- Next smallest unvisited element is 2:
$$2 \overset{\tau}{\mapsto} 5 \overset{\tau}{\mapsto} 2.$$
This gives the second cycle: $(2\ 5)$. Mark 2 and 5 as visited.
- The last unvisited element is 3, and since
$$3 \overset{\tau}{\mapsto} 3,$$
this is a fixed point (a 1-cycle), written as (3).

Combining the above, the full cycle decomposition is:

$$\tau = (1\ 4)(2\ 5)(3). \tag{1.19}$$

Note that the *order* in which disjoint cycles are written is irrelevant. For instance, in Example 1.2.4, all of the following represent the same permutation:

$$\tau = (1\ 4)(2\ 5)(3) = (2\ 5)(1\ 4)(3) = (3)(1\ 4)(2\ 5).$$

When the total number of elements k is clear from the context (e.g., $k = 5$ in this case), it is common to omit one-cycles, also denoted *fixed points*, because these elements are understood to remain unchanged under the permutation. Using this convention, the permutation in Example 1.2.4 is simply written as:

$$\tau = (1\ 4)(2\ 5). \tag{1.20}$$

With this notation, the identity permutation is denoted by $\iota = ()$, which is shorthand for $\iota = (1)(2)\cdots(k)$ – that is, all elements are fixed.

The *cycle structure* of a permutation, denoted by $\lambda(\pi)$, is the list of the lengths of its disjoint cycles. For example, the permutation $\tau = (1\ 4)(2\ 5)$ has cycle structure $\lambda(\tau) = (2,2,1)$. The number of disjoint cycles is then given by the length of this list:

$$\#(\pi) = |\lambda(\pi)|, \qquad \text{for any } \pi \in \mathrm{S}_k\,. \tag{1.21}$$

Example 1.2.5: Cycle Structure

Consider the following permutations of 6 elements

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 1 & 6 & 3 & 2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 4 & 1 & 3 \end{pmatrix}. \tag{1.22}$$

It is a simple exercise to show that their cycle decomposition is $\tau = (1,5,3)(2,4,6)$ and $\sigma = (1,5)(3,6)$ [which is a shorthand notation for $\sigma = (1,5)(3,6)(2)(4)$]. The permutation τ has two cycles of length 3, hence the cycle structure is $\lambda(\tau) = (3,3)$. Instead, σ is composed of two 2-cycles and two 1-cycles, so the cycle structure is $\lambda(\sigma) = (2,2,1,1)$.

Exercise 1.3. *Show that conjugation preserves the cycle structure of a permutation. That is, for any $\sigma, \pi \in \mathrm{S}_k$, prove that*

$$\lambda(\pi\sigma\pi^{-1}) = \lambda(\sigma).$$

Transpositions, also known as *swaps*, are a special class of permutations that exchange exactly two elements and leave all others unchanged. A transposition has the form:

$$\sigma = (i\ j) = \begin{pmatrix} 1 & \dots & i & \dots & j & \dots & k \\ 1 & \dots & j & \dots & i & \dots & k \end{pmatrix}. \tag{1.23}$$

It is straightforward to verify that *any permutation* can be written as a product of transpositions. However, unlike the decomposition into disjoint cycles, this representation is *not unique* – a given permutation can be written in many different ways as a product of transpositions. While this makes the transposition decomposition less suitable for labeling permutations, it is extremely useful in algebraic manipulations. For instance, it turns out that the number of transpositions in any representation is always odd or always even, which justifies the definition of the *sign of a permutation*:

$$\mathrm{sgn}(\sigma) = (-1)^{\#\text{transpositions in } \sigma}\,. \tag{1.24}$$

The sign function is important for the construction of representations of S_k and plays an important role in multilinear algebra, in particular in the definition of the *determinant of a matrix*.

We conclude this section by reviewing some structural aspects of the symmetric groups S_k for varying values of k. A key property is that the group S_k naturally embeds into S_{k+1}: that is, every permutation of k elements can be viewed as a permutation of $k+1$ elements that leaves the $(k+1)$-th element fixed. More formally, we can write

$$\mathrm{S}_{k+1} = \mathrm{S}_k \sqcup \{(j\ k+1)\cdot\sigma\ :\ \sigma \in \mathrm{S}_k,\ j = 1,2,\dots,k\}\,, \tag{1.25}$$

where the union is disjoint and the second term represents all permutations obtained by composing an element of S_k with a transposition that swaps $k+1$ with one of the first k elements.

This recursive structure is useful for establishing many properties of permutations by induction on k. A simple but important example is the total number of elements in the symmetric group.

Theorem 1.1 (Counting of Permutations)**.** *Given $k \geq 1$, the total number of permutations in S_k is*

$$|\,\mathrm{S}_k\,| = k!. \tag{1.26}$$

Proof. The total number of permutations is given by the factorial $k! = k(k-1)\cdots 2\cdot 1$, with the convention that $0! = 1$. To prove this, we use induction.

For $k = 1$, the symmetric group $\mathrm{S}_1 = \{\iota\}$ consists only of the identity permutation, so $|\,\mathrm{S}_1\,| = 1 = 1!$.

Assume now that $|\,\mathrm{S}_k\,| = k!$ for some $k \geq 1$. From Eq. (1.25), the next symmetric group can be written as

$$\mathrm{S}_{k+1} = \mathrm{S}_k \sqcup \{(j\ k+1)\cdot\sigma :\ \sigma \in \mathrm{S}_k,\ j = 1,\ldots,k\}\,. \tag{1.27}$$

There are k possible values of j, and for each j, σ runs over all $k!$ permutations in S_k. Hence,

$$|\,\mathrm{S}_{k+1}\,| = |\,\mathrm{S}_k\,| + k\cdot|\,\mathrm{S}_k\,| = k! + k\cdot k! = (k+1)\cdot k! = (k+1)!. \tag{1.28}$$

This completes the proof by induction. □

Consider now the following permutations of six elements:

$$\sigma_1 = (1\ 2\ 3)(5\ 6), \quad \sigma_2 = (2\ 4\ 6\ 1), \quad \sigma_3 = (1\ 2)(3\ 4)(5\ 6). \tag{1.29}$$

While their cycle structures differ, all three permutations have exactly three disjoint cycles. This illustrates that simply counting the number of cycles is a coarser classification than specifying the full *cycle structure*.

In many computations throughout this course, we will be interested in the number of permutations in S_k with a fixed number of cycles $r = \#(\sigma)$. This quantity is given by the *unsigned Stirling numbers of the first kind*, denoted by $c(k,r)$. These numbers satisfy the recursive relation:

$$c(k+1,r) = k\cdot c(k,r) + c(k,r-1), \tag{1.30}$$

which allows them to be computed inductively, without explicitly listing all permutations. These numbers form a triangle similar to Pascal's triangle and are tabulated up to $k = 10$ in the Appendix.

Starting with the base case $k = 1$, where $\mathrm{S}_1 = \{()\}$, we find:

$$c(1,0) = 0, \qquad c(1,1) = 1. \tag{1.31}$$

Using the recurrence, the next values for $k = 2$ are:

$$c(2,0) = 0, \qquad c(2,1) = 1, \qquad c(2,2) = 1. \tag{1.32}$$

Exercise 1.4. *Compute the values $c(k,r)$ for $1 \leq r \leq k$ when $k = 3$ and $k = 4$. Verify that:*

$$\sum_{r=1}^{k} c(k,r) = k!\,, \qquad\qquad \sum_{r=1}^{k} c(k,r)x^r = x(x+1)\cdots(x+k-1)\,. \tag{1.33}$$

The first identity reflects the fact that summing over all permutations with a fixed number of cycles r recovers the total number of permutations in S_k, while the second gives another combinatorial interpretation of $c(k,r)$ as the coefficients in the power series of the 'rising factorial'.

1.2.2 The Action of Permutations on Quantum States

Throughout this course, we work with a d-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$, equipped with the standard orthonormal basis $\{|x\rangle\}_{x=0}^{d-1}$. Our main object of interest is the k-fold tensor product space $\mathcal{H}^{\otimes k} = (\mathbb{C}^d)^{\otimes k}$, often referred to as the *replica space* in the many-body literature.

Elements of the symmetric group S_k act naturally on this space by permuting the tensor factors. Concretely, given a permutation $\sigma \in \mathrm{S}_k$, its action on a product basis state is defined as:

$$R_\sigma|x_1, x_2, \ldots, x_k\rangle = |x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}\rangle\,. \tag{1.34}$$

Consider an example with $k = 4$ and $\sigma = (1\ 4\ 3)(2) \in S_4$. We have

$$R_\sigma |x_1, x_2, x_3, x_4\rangle = |x_4, x_2, x_1, x_3\rangle . \tag{1.35}$$

Equation (1.34) defines a linear (even unitary) operator R_σ on $\mathcal{H}^{\otimes k}$. The map

$$R\colon S_k \longrightarrow U(\mathcal{H}^{\otimes k}), \quad \sigma \mapsto R_\sigma \tag{1.36}$$

is a so-called *group representation* of S_k. This means it respects the group structure of S_k, more specifically:

$$R_{\sigma\pi} = R_\pi R_\sigma, \qquad R_\iota = \mathbb{1}, \qquad R_{\sigma^{-1}} = R_\sigma^{-1}, \qquad R_\sigma^\dagger = R_\sigma^{-1}. \tag{1.37}$$

These properties follow directly from the definition in Eq. (1.34). As an explicit verification of the composition law, let us define $\tilde{x}_i := x_{\sigma(i)}$ and compute:

$$R_\pi R_\sigma |x_1, x_2, \ldots, x_k\rangle = R_\pi |x_{\sigma(1)}, \ldots, x_{\sigma(k)}\rangle = |\tilde{x}_{\pi(1)}, \ldots, \tilde{x}_{\pi(k)}\rangle \tag{1.38}$$

$$= |x_{(\sigma\cdot\pi)(1)}, \ldots, x_{(\sigma\cdot\pi)(k)}\rangle = R_{\sigma\cdot\pi}|x_1, x_2, \ldots, x_k\rangle. \tag{1.39}$$

We leave the verification of the other axioms – such as unitarity and inverse consistency – as an exercise.

Exercise 1.5. *Verify that R defines a unitary representation of the symmetric group S_k, i.e., check Eq. (1.37).*

Exercise 1.6. *Show that permutations act on tensor products of operators as follows:*

$$R_\sigma (A_1 \otimes A_2 \otimes \cdots \otimes A_k) R_\sigma^\dagger = A_{\sigma(1)} \otimes A_{\sigma(2)} \otimes \cdots \otimes A_{\sigma(k)}. \tag{1.40}$$

Hint: *Apply both sides to a product basis state.*

Composite Systems In practice, we often work with *multi-qudit systems* described by a Hilbert space of the form $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$, corresponding to n qudits of local dimension q. In this setting, the k-fold copy of the system is given by the tensor product

$$((\mathbb{C}^q)^{\otimes n})^{\otimes k}, \tag{1.41}$$

which can be naturally visualized as a $k \times n$ grid of qudits (see Fig. 1.1). Each row represents one replica of the full system, and each column represents the k copies of a single local qudit.

Quantum operations such as global unitaries typically act *row-wise*, that is, identically and independently on each copy. Such operations are of the form

$$U^{\otimes k}, \qquad \text{where } U \in U((\mathbb{C}^q)^{\otimes n}), \tag{1.42}$$

meaning that the unitary acts in parallel across the k rows.

In contrast, permutations act by permuting the *rows*, i.e., the k copies of each local qudit. This operation is performed *column-wise*, and can be implemented by applying the same permutation operator to each column in parallel. This leads to a convenient factorized structure: if we reinterpret the total space via the isomorphism

$$((\mathbb{C}^q)^{\otimes n})^{\otimes k} \simeq ((\mathbb{C}^q)^{\otimes k})^{\otimes n}, \tag{1.43}$$

then the permutation operator R_π acting on the full system decomposes as

$$R_\pi = r_\pi^{\otimes n}, \tag{1.44}$$

where r_π acts on the k-dimensional replica space associated with each local qudit. This "horizontal" factorization is exactly what is depicted in Fig. 1.1 and will be essential in constructing efficient representations of randomized operations throughout the course.

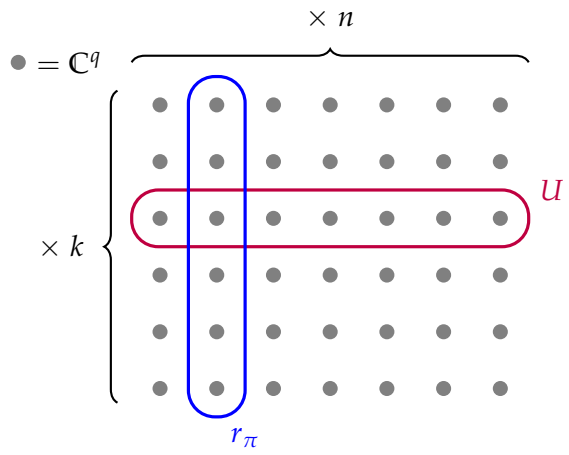


Figure 1.1: The Hilbert space $((\mathbb{C}^q)^{\otimes n})^{\otimes k}$ depicted as a $k \times n$ grid where every point corresponds to a copy of $\mathbb{C}^q$. Unitaries $U \in \mathrm{U}(q^n)$ act row-wise on the grid, while permutations $\pi \in \mathrm{S}_k$ act column-wise.

Traces Traces will play a fundamental role for explicit computations. Here, we derive a formula for the trace of permutation operators and for the trace of a product operator multiplied with a permutation. Let us first consider the cyclic permutation $\gamma = (1, 2, \ldots, k)$. Then, we compute

$$\begin{aligned}\mathrm{tr}(R_\gamma) &= \sum_{x_1,\ldots,x_k=0}^{d-1} \langle x_1, \ldots, x_k | R_\gamma | x_1, \ldots, x_k \rangle \\ &= \sum_{x_1,\ldots,x_k} \langle x_1, \ldots, x_k | x_2, \ldots, x_k, x_1 \rangle = \sum_{x_1,\ldots,x_k} \delta_{x_1,x_2}\delta_{x_2,x_3}\ldots\delta_{x_{k-1},x_k}\delta_{x_k,x_1} = d. \quad (1.45)\end{aligned}$$

Next, consider an arbitrary permutation σ. Then, we can find a permutation π such that $\pi\sigma\pi^{-1} = (1, \ldots, b_1)(b_1+1, b_1+2, \ldots, b_2)\cdots(b_{r-1}+1, b_{r-1}+2, \ldots, b_r)$ where $r = \#\sigma$ is the number of cycles in σ (recall that the cycle structures of σ and $\pi\sigma\pi^{-1}$ have to necessarily match, cf. Ex. 1.3). But $R_{\pi\sigma\pi^{-1}}$ is simply a tensor product of cyclic permutations on $\mathcal{H}^{\otimes b_1}, \ldots, \mathcal{H}^{\otimes(b_2-b_1)}, \ldots, \mathcal{H}^{\otimes(b_r-b_{r-1})}$ and thus

$$\mathrm{tr}(R_\sigma) = \mathrm{tr}(R_{\pi\sigma\pi^{-1}}) = \prod_{i=1}^{r} d = d^r = d^{\#\sigma}. \quad (1.46)$$

Beyond this simple situation, we often need to compute the trace of product operators multiplied with a permutation operator, i.e. an expression of the form $\mathrm{tr}(A_1 \otimes A_2 \otimes \cdots \otimes A_k R_\sigma)$. Here, we can follow the same arguments as above: First, if $\sigma = \gamma = (1, \ldots, k)$ is the cyclic permutation, then

$$\begin{aligned}\mathrm{tr}(A_1 \otimes A_2 \otimes \cdots \otimes A_k R_\gamma) &= \sum_{x_1,\ldots,x_k} \langle x_1, \ldots, x_k | A_1 \otimes A_2 \otimes \cdots \otimes A_k | x_2, x_3, \ldots, x_k, x_1 \rangle \quad (1.47)\\ &= \sum_{x_1,\ldots,x_k} (A_1)_{x_1,x_2}(A_2)_{x_2,x_3}\cdots(A_k)_{x_k,x_1} = \mathrm{tr}(A_1 A_2 \cdots A_k). \quad (1.48)\end{aligned}$$

Next, for an arbitrary $\sigma = (a_1, \ldots, a_{j_1})(a_{j_1+1}, \ldots, a_{j_2})\ldots(a_{j_{r-1}+1}, \ldots, a_k)$, find again a permutation π that 'orders the cycles' as $\pi\sigma\pi^{-1} = (1, \ldots, b_1)(b_1+1, b_1+2, \ldots, b_2)\cdots(b_{r-1}+1, b_{r-1}+2, \ldots, b_r)$ and then use Ex. 1.6 to conclude that

$$\begin{aligned}\mathrm{tr}(A_1 \otimes A_2 \otimes \cdots \otimes A_k R_\sigma) &= \mathrm{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(k)} R_{\pi\sigma\pi^{-1}}) \quad (1.49)\\ &= \mathrm{tr}(A_{\pi(1)} \cdots A_{\pi(b_1)}) \cdots \mathrm{tr}(A_{\pi(b_{r-1}+1)} \cdots A_{\pi(b_r)}) \quad (1.50)\\ &= \mathrm{tr}(A_{a_1} \cdots A_{a_{j_1}}) \cdots \mathrm{tr}(A_{a_{j_{r-1}+1}} \cdots A_{a_k}). \quad (1.51)\end{aligned}$$

For the important case when $A_1 = A_2 = \cdots = A_k$, the final result is simplified to

$$\mathrm{tr}(A^{\otimes k} R_\sigma) = \prod_{c \in \lambda(\sigma)} \mathrm{tr}(A^c), \quad (1.52)$$

where $\lambda(\sigma)$ is the cycle structure of the permutation σ.

Symmetric subspace Throughout this course, symmetries under permutations will be a fundamental role, in particular the subspace of $(\mathbb{C}^d)^{\otimes k}$ composed of vectors that are left invariant by permutations:

$$\mathrm{Sym}_{k,d} \equiv \mathrm{Sym}((\mathbb{C}^d)^{\otimes k}) := \{\psi \in (\mathbb{C}^d)^{\otimes k} \mid R_\sigma|\psi\rangle = |\psi\rangle \ \forall \sigma \in \mathrm{S}_k\}\,. \tag{1.53}$$

We will now show that the projector onto $\mathrm{Sym}_{k,d}$ is

$$P_{\mathrm{Sym},k,d} = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k} R_\sigma\,. \tag{1.54}$$

To see this, we first check that $P_{\mathrm{Sym},k,d}$ is an orthogonal projector:

$$P^2_{\mathrm{Sym},k,d} = \frac{1}{(k!)^2}\sum_{\sigma,\pi\in\mathrm{S}_k} R_{\sigma\pi}\,. = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k}\frac{1}{k!}\sum_{\tau\in\mathrm{S}_k} R_\tau = P_{\mathrm{Sym},k,d}\,, \tag{1.55}$$

$$P^\dagger_{\mathrm{Sym},k,d} = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k} R_{\sigma^{-1}} = \frac{1}{k!}\sum_{\pi\in\mathrm{S}_k} R_\pi = P_{\mathrm{Sym},k,d}\,, \tag{1.56}$$

where we substituted variables as $\tau = \sigma\pi$ and $\pi = \sigma^{-1}$, respectively, and used that the sum is invariant under the change of variables. Next, note that for all $\psi \in \mathrm{Sym}_{k,d}$:

$$P_{\mathrm{Sym},k,d}|\psi\rangle = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k} R_\sigma|\psi\rangle = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k}|\psi\rangle = |\psi\rangle\,, \tag{1.57}$$

thus, $\mathrm{Sym}_{k,d}$ is in the range of $P_{\mathrm{Sym},k,d}$. Moreover, if $P_{\mathrm{Sym},k,d}|\psi\rangle = |\psi\rangle$, then

$$R_\pi|\psi\rangle = R_\pi P_{\mathrm{Sym},k,d}|\psi\rangle = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k} R_{\pi\sigma}|\psi\rangle = \frac{1}{k!}\sum_{\tau\in\mathrm{S}_k} R_\tau|\psi\rangle = P_{\mathrm{Sym},k,d}|\psi\rangle = |\psi\rangle\,, \tag{1.58}$$

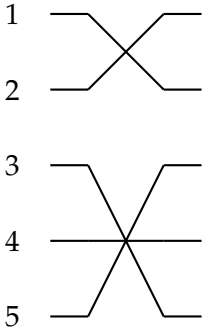
and thus the range of $P_{\mathrm{Sym},k,d}$ is exactly $\mathrm{Sym}_{k,d}$. We can now compute the dimension of the symmetric subspace using Eq. (1.46) and Ex. 1.4 as

$$\dim \mathrm{Sym}_{k,d} = \mathrm{tr}\, P_{\mathrm{Sym},k,d} = \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k} d^{\#\sigma} = \frac{1}{k!}\sum_{r=1}^{k} c(k,r)d^r = \frac{d(d+1)\cdots(d+k-1)}{k!} = \binom{d+k-1}{k}\,. \tag{1.59}$$

Graphical representation While the above methodologies are generic and straightforward, the algebra is often cumbersome. For this reason, it is useful to introduce a graphical notation to represent permutation. Similar to the Feynman diagrammatics for perturbative expansions, this is simply a bookkeeping of the operation previously described.

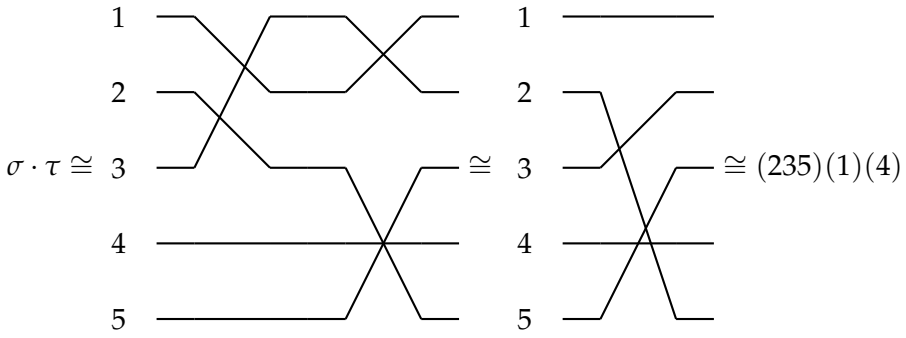
We denote permutations as lines connecting the list $\{1,2,\ldots,k\}$ to the output $\{\sigma(1),\sigma(2),\ldots,\sigma(k)\}$. For example, given $\tau = (12)(35)(4)$ a 5 elements permutation, we can represent it as

$(12)(35)(4) \cong$



(1.60)

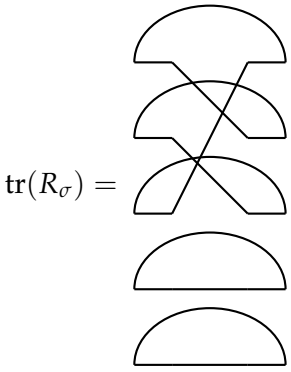
This notation is particularly useful, since it makes computing products particularly easy. For example, the product of τ with $\sigma = (123)(4)(5)$, we simply need to follow the lines after connecting them, specifically



(1.61)

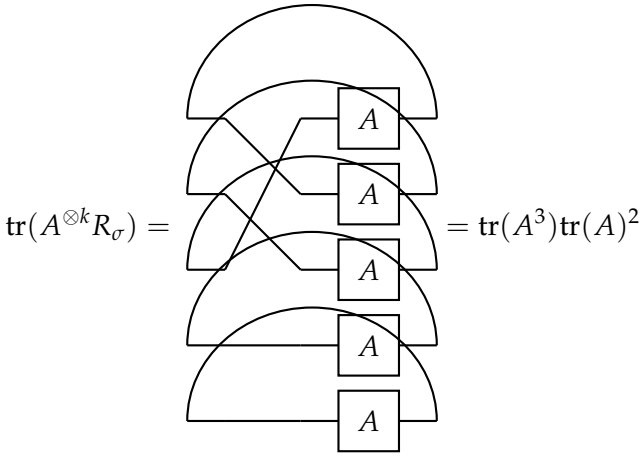
Since the group structure is the same for the representations of S_k, we can use the same notation also for the operators $\{R_\sigma : \sigma \in S_k\}$. For representations, the diagrammatic notation allows also to include traces and product by operators.

Traces require to add a curve that connect initial and final endpoints. For example:



(1.62)

Similarly, for operators we have



(1.63)

CHAPTER 2

QUANTUM RANDOMNESS I

In this section, we lay the foundation for one of the central objects in quantum information theory and random quantum systems: the *Haar twirl*. This is the average over the unitary group of a replicated quantum channel, defined as

$$\mathsf{M}_k(A) := \int_{\mathrm{U}(d)} U^{\otimes k} A U^{\otimes k,\dagger} \mathrm{d}U \,. \tag{2.1}$$

This map, known as the k-th unitary twirl, appears repeatedly throughout these notes and plays a fundamental role in both quantum information theory and many-body physics. It arises in contexts ranging from randomized benchmarking and quantum designs to entanglement theory, thermalization, and quantum chaos.

We begin by briefly discussing the structure of the Haar measure dU on the unitary group $\mathrm{U}(d)$, and how random unitaries can be parametrized in practice. We then turn to the evaluation of the Haar twirl integral (2.1). The key observation is that $\mathsf{M}_k(A)$ lies in the so-called *commutant* of $U^{\otimes k}$, i.e., the space of operators that commute with all $U^{\otimes k}$ for $U \in \mathrm{U}(d)$. A powerful result from representation theory – the *Schur-Weyl duality* – tells us that this commutant is spanned by the action of the symmetric group via permutations on $\mathcal{H}^{\otimes k}$.

This insight allows us to express the Haar twirl as a linear combination over permutations, with coefficients given by the *Weingarten calculus* – a systematic method to evaluate integrals over the unitary group. The remainder of this section is devoted to developing these tools and applying them to compute $\mathsf{M}_k(A)$ explicitly.

2.1 What is a Haar measure?

On the real line $\mathbb{R}$, there is a unique measure dx satisfying two simple but fundamental properties

$$\mathrm{d}(x+y) = \mathrm{d}x \qquad \text{(translation invariance)}\,, \qquad\qquad \int_0^1 \mathrm{d}x = 1 \qquad \text{(unit volume)}\,. \tag{2.2}$$

This measure represents what we intuitively mean by a *uniform* distribution: translation invariance ensures that no point is preferred over any other. However, since $\mathbb{R}$ is non-compact, this measure assigns infinite volume to the full space, and thus cannot be normalized to a probability measure. To work around this, we typically restrict to a compact interval such as $[0,1]$, which provides a natural normalization.

Remarkably, this idea of defining an invariant, uniform measure generalizes to far more abstract settings. Whenever the underlying set carries a suitable group structure, one can often define a natural measure that is invariant under group multiplication. More precisely, if G is a compact (Hausdorff topological) group then there exists a unique measure dg on G that is both

$$\text{left/right invariant} \qquad \mathrm{d}g = \mathrm{d}(hg) = \mathrm{d}(gh) \tag{2.3}$$

$$\text{normalized} \qquad \int_G \mathrm{d}g = 1\,. \tag{2.4}$$

This unique measure is called the *Haar (or uniform) measure* on G.

In this course, our primary interest lies in the unitary group $\mathrm{U}(d)$, which is compact and satisfied the condition above. Concretely, the Haar measure dU on $\mathrm{U}(d)$ satisfies

$$\mathrm{d}U = \mathrm{d}(UV) = \mathrm{d}(VU) \qquad \int_{\mathrm{U}(d)} \mathrm{d}U = 1\,. \tag{2.5}$$

This measure provides a rigorous definition of what it means to sample a unitary matrix uniformly at random. In principle, one could perform such an integration by explicitly parametrizing the unitaries and integrating over the associated coordinates. However, this approach becomes highly impractical as the dimension d increases, due to the complicated geometry and rapidly growing number of parameters. Instead, we will introduce more powerful techniques—based on symmetry and representation theory—that allow us to compute Haar integrals in an efficient and conceptually clean way.

Example 2.1.1: Cycle Structure

Consider $d = 2$ and let us compute the Haar average

$$\int_{\mathrm{U}(2)} \mathrm{d}U\, UAU^\dagger \tag{2.6}$$

for a general operator

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} . \tag{2.7}$$

A generic unitary on $\mathrm{U}(2)$ is given by

$$U(\theta, \phi, \psi) = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{i\psi}\sin\theta & e^{i(\phi+\psi)}\cos\theta \end{pmatrix} \tag{2.8}$$

with the Haar measure $\mathrm{d}U = (4\pi^2)^{-1}\sin(2\theta)\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi$ with $\theta \in [0, \pi/2]$ and $\phi, \psi \in [0, 2\pi]$. (We ignore the global phase since it cancels trivially in $UAU^\dagger$. The integral acts on each element of the total matrix $(UAU^\dagger)_{ij} = \sum_{k,l} U_{i,k}A_{k,l}\bar{U}_{j,l}$. For concreteness, let us compute only the entry $i = j = 1$: $(U^\dagger AU)_{11} = u_{11}a_{11}\bar{u}_{11} + u_{11}a_{12}\bar{u}_{12} + u_{12}a_{21}\bar{u}_{11} + u_{22}a_{22}\bar{u}_{22}$. Using the well known formula $\int \mathrm{d}\alpha e^{in\alpha} = \delta_{n,0}$, we find that the terms a_{12} and a_{21} simplify. The remaining computation requires

$$\int_0^{\pi/2} \mathrm{d}\theta \sin(2\theta)(\cos^2\theta a_{11} + \sin^2\theta a_{22}) = \frac{1}{2}(a_{11} + a_{22}). \tag{2.9}$$

Working out the details for the other indices, we obtain

$$\int_{\mathrm{U}(2)} \mathrm{d}U\, UAU^\dagger = \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & 0 \\ 0 & a_{11}+a_{22} \end{pmatrix} = \frac{1}{2}\mathrm{tr}(A)\mathbb{I}. \tag{2.10}$$

2.1.1 The unitary commutant

In this section, we study the subspace of operators that commute with $U^{\otimes k}$, which we call the (k-fold) commutant of $\mathrm{U}(d)$:

$$\mathrm{Comm}_k = \{A \in \mathrm{L}(\mathcal{H}^{\otimes k}) \,|\, U^{\otimes k}A = AU^{\otimes k}\, \forall U \in \mathrm{U}(d)\}\,. \tag{2.11}$$

We care about this because of the following, elementary observation:

Lemma 2.1. $\mathsf{M}_k(A) \in \mathrm{Comm}_k$ *for all* $A \in \mathrm{L}(\mathcal{H}^{\otimes k})$. *In fact, every element in* Comm_k *is of the form* $\mathsf{M}_k(A)$.

Proof. Clearly, if A commutes with all $U^{\otimes k}$, then $\mathsf{M}_k(A) = A$, i.e. every element in Comm_k is of the form $\mathsf{M}_k(A)$. Vice versa, if $B = \mathsf{M}_k(A)$, then, by the invariance of the Haar measure:

$$U^{\otimes k}BU^{\otimes k,\dagger} = \int_{\mathrm{U}(d)} (UV)^{\otimes k}A(UV)^{\otimes k,\dagger}\mathrm{d}V = \int_{\mathrm{U}(d)} V^{\otimes k}AV^{\otimes k,\dagger}\mathrm{d}V = B\,, \quad \forall U \in \mathrm{U}(d)\,. \tag{2.12}$$

Thus, $B \in \mathrm{Comm}_k$. □

A central step in understanding the commutant is to note that the representation of S_k on $\mathcal{H}^{\otimes k}$ introduced in Sec. 1.2.2, this is $R_\pi|x_1,\dots,x_k\rangle = |x_{\pi(1)},\dots,x_{\pi(k)}\rangle$, clearly commutes with $U^{\otimes k}$. In representation-theoretic terms, the representations of $U(d)$ and S_k are said to be *dual to each other*. This implies that the permutations are contained in the unitary commutant, $R_\pi \in \mathrm{Comm}_k$ for all $\pi \in S_k$. A fundamental result in representation theory, **Schur-Weyl duality**, even states that every element in the commutant is a linear combination of permutations. We will only state this result here and refer for a proof to the literature **[tbd]**.

Theorem 2.1 (Schur-Weyl duality)**.** *The k-fold unitary commutant* Comm_k *is spanned by* $\{R_\sigma \,|\, \sigma \in S_k\}$. *Vice versa, the commutant of* $\{R_\sigma \,|\, \sigma \in S_k\}$ *is spanned by* $\{U^{\otimes k} \,|\, U \in U(d)\}$

A natural question to ask is whether the permutations form a basis for the commutant. Intriguingly, this is the case if the dimension d is large enough:

Lemma 2.2. *The set* $\{R_\sigma \,|\, \sigma \in S_k\}$ *is linearly independent for* $d \geq k$.

Proof. We consider the standard basis of $\mathbb{C}^d$, which we here denote as $|1\rangle,\dots,|d\rangle$. Since $k \leq d$, we can consider the action of permutations on $|1,\dots,k\rangle \in (\mathbb{C}^d)^{\otimes k}$:

$$R(\pi)|1,\dots,k\rangle = |\pi(1),\dots,\pi(k)\rangle\,. \tag{2.13}$$

Now, if $R(\pi)$ and $R(\sigma)$ would be linearly dependent ($\pi \neq \sigma$), than so would be the states $|\pi(1),\dots,\pi(k)\rangle$ and $|\sigma(1),\dots,\sigma(k)\rangle$. However, these are distinct elements from a basis, thus we arrive at a contradiction. □

Remark 2.1. *In fact, permutations become linearly dependent as soon as* $k > d$. *We do not need this statement in the following, thus we will not treat the proof in the lecture. We however state it here for completeness. We consider the* antisymmetric subspace $\mathrm{Alt}_{k,d} \subset (\mathbb{C}^d)^{\otimes k}$ *which is the joint* -1 *eigenspace of all* transpositions $R((ij))$ *for* $(ij) \in S_k$. *The projector onto* $\mathrm{Alt}_{k,d}$ *has the general form*

$$P_{\mathrm{Alt},k,d} = \frac{1}{k!}\sum_{\pi\in S_k} \mathrm{sgn}(\pi) R(\pi)\,, \tag{2.14}$$

where the sign function $\mathrm{sgn}(\pi)$ *is given as follows: Decompose* π *into transpositions only, then count the number of transpositions needed. If it is even,* $\mathrm{sgn}(\pi) = 1$, *else* $\mathrm{sgn}(\pi) = -1$. *Now,* $\dim \mathrm{Alt}_{k,d} = \binom{d}{k} = 0$ *if* $k > d$, *and hence* $P_{\mathrm{Alt},k,d} = 0$. *This gives a non-trivial linear relation between permutations, i.e. they are linearly dependent.*

2.1.2 Weingarten calculus

By the previous findings, we can always write $M_k(A)$ as a linear combination

$$M_k(A) = \sum_{\pi\in S_k} c_\pi(A) R_\pi\,, \tag{2.15}$$

for some coefficients $c_\pi(A)$. Note that taking the trace inner product of $M_k(A)$ with a fixed permutation $R_\sigma^\dagger$ yields

$$\mathrm{tr}(R_\sigma^\dagger M_k(A)) = \int \mathrm{tr}(R_\sigma^\dagger U^{\otimes k} A U^{\otimes k,\dagger}) dU = \int \mathrm{tr}(U^{\otimes k,\dagger} R_\sigma^\dagger U^{\otimes k} A) dU = \mathrm{tr}(R_\sigma^\dagger A)\,. \tag{2.16}$$

However, we also have

$$\mathrm{tr}(R_\sigma^\dagger A) = \mathrm{tr}(R_\sigma^\dagger M_k(A)) = \sum_{\pi\in S_k} c_\pi(A)\,\mathrm{tr}(R_\sigma^\dagger R_\pi) =: \sum_{\pi\in S_k} c_\pi(A) G_{\sigma,\pi}\,, \tag{2.17}$$

where we defined the *Gram matrix*

$$G_{\pi,\sigma} := \mathrm{tr}(R_\pi^\dagger R_\sigma) = \mathrm{tr}(R_\pi^\dagger R_\sigma) = \mathrm{tr}(R_{\pi^{-1}\sigma}) = d^{\#(\pi^{-1}\sigma)}\,. \tag{2.18}$$

Here, we used that R is a representation to combine the product of permutation operators, and the trace formula (1.46). Setting $a_\sigma := \operatorname{tr}(R_\sigma^\dagger A)$, we can write the above equation in matrix form as $a = Gc$, which we could hope to invert to get an expression for the coefficient vector c. Note that the permutations are not orthogonal with respect to the trace inner product $(A|B) = \operatorname{tr}(A^\dagger B)$, and hence the Gram matrix is not simply diagonal. However, we know that the permutations span the commutant and that $\mathsf{M}_k(A)$ lies in the commutant. Hence, the equation $a = Gc$ always has a solution and it is unique if and only if the permutations form a basis, i.e. iff $d \geq k$, which is what we will always assume for the remainder of this course.[1] Then, this solution is simply $c = G^{-1}a$, or put differently,

$$\mathsf{M}_k(A) = \sum_{\pi,\sigma\in\mathsf{S}_k} W_{\pi,\sigma} \operatorname{tr}(R_\sigma^\dagger A) R_\pi \,, \tag{2.19}$$

where we defined $W := G^{-1}$, the so-called *Weingarten matrix*. Knowing the Weingarten matrix allows us to compute integrals of the form (2.1) using the Weingarten expansion (2.19).

Properties of the Gram and Weingarten matrix The Gram and Weingarten matrix have a substantial structure which directly relates to the representation theory of the symmetric group. We will not dive into these details in this course, but instead prove some concrete relations. We summarize them in the following.

Lemma 2.3. *The Gram and Weingarten matrix fulfill the following properties.*

(a) $G_{\pi,\sigma}$ and $W_{\pi,\sigma}$ only depend on $\pi^{-1}\sigma$.

(b) The row and column sums of G are constant:

$$\mathcal{G}_{k,d} := \sum_\sigma G_{\pi,\sigma} = \sum_\pi G_{\pi,\sigma} = \frac{(d+k-1)!}{(d-1)!} = d(d+1)\cdots(d+k-1)\,. \tag{2.20}$$

(c) The row and column sums of W are constant:

$$\sum_\sigma W_{\pi,\sigma} = \sum_\pi W_{\pi,\sigma} = \mathcal{G}_{k,d}^{-1} = \frac{(d-1)!}{(d+k-1)!}\,. \tag{2.21}$$

Proof. (a) Clearly, $G_{\pi,\sigma}$ depends only on $\pi^{-1}\sigma$ by definition (cf. Eq. (2.18)). Now note that this implies that G is invariant under simultaneous row and column permutations. Indeed, if T_τ is the permutation matrix acting as $T_\tau|e_\pi\rangle = |e_{\tau\pi}\rangle$, then $(T_\tau^{-1}GT_\tau)_{\sigma,\pi} = G_{\tau\sigma,\tau\pi} = G_{\sigma,\tau}$. Inverting $G = T_\tau^{-1}GT_\tau$ yields $W = T_\tau^{-1}WT_\tau$ and thus $W_{\pi,\sigma} = W_{\tau\pi,\tau\sigma}$ for all τ, in particular $W_{\pi,\sigma} = W_{\mathrm{id},\pi^{-1}\sigma}$ for $\tau = \pi^{-1}$. For (b), we compute

$$\mathcal{G}_{k,d} = \sum_\sigma \operatorname{tr}(R_{\pi^{-1}\sigma}) = \sum_\sigma \operatorname{tr}(R_\sigma) = k!\operatorname{tr}(P_{\mathrm{Sym},k,d}) = k!\binom{d+k-1}{k} = \frac{(d+k-1)!}{(d-1)!}\,. \tag{2.22}$$

Here, we used that the multiplication by π^{-1} can be absorbed into the sum (variable change), and the definition of the projector onto the symmetric subspace, $P_{\mathrm{Sym},k,d} = \frac{1}{k!}\sum_\sigma R_\sigma$, and the value of its trace, cf. Eqs. (1.54) and (1.59). For (c), we note that the definition of W as inverse of G implies

$$\sum_\pi W_{\sigma,\pi}G_{\pi,\tau} = \delta_{\sigma,\tau} \quad \Rightarrow \quad 1 = \sum_{\pi,\tau} W_{\sigma,\pi}G_{\pi,\tau} = \mathcal{G}_{k,d}\sum_\pi W_{\sigma,\pi} \quad \Rightarrow \quad \sum_\pi W_{\sigma,\pi} = \mathcal{G}_{k,d}^{-1}\,. \tag{2.23}$$

□

[1] It is however not terribly complicated to make this work for $d < k$, see Sec. 2.3.

Some examples and exercises In the following, we will compute the Weingarten matrix for small values of k and illustrate the computations of Haar integrals using a number of examples. To this end, we use that the Gram matrix has a very simple form: It is the trace of a permutation $R_\tau = R_\pi^\dagger R_\sigma$ and we gave an expression for this in Eq. (1.46).

Example 2.1.2: Weingarten matrix for $k = 2$

Let us consider $k = 2$. Then we only have two permutations: the identity $\mathbb{1}$ and the flip/swap $\mathbb{F} = (2\,1)$. There is only one non-trivial matrix element, namely $G_{\mathbb{1},\mathbb{F}} = \mathrm{tr}(\mathbb{F}) = d$. Hence, the Gram and Weingarten matrices are

$$G = d^2 \begin{pmatrix} 1 & d^{-1} \\ d^{-1} & 1 \end{pmatrix}, \qquad W = \frac{1}{d^2-1} \begin{pmatrix} 1 & -d^{-1} \\ -d^{-1} & 1 \end{pmatrix}. \tag{2.24}$$

Example 2.1.3: Average collision probability

The probability of obtaining the computational basis state x on $U|0\rangle$ is $p(x|U) = |\langle x\,|U|0\rangle|^2$. A measure of flatness of this distribution is the *collision probability*:

$$Z_U := \sum_x p(x|U)^2 = \sum_x |\langle x\,|U|0\rangle|^4 . \tag{2.25}$$

Here, we are interested on how flat this distribution is *on average*, over Haar-random unitaries U. Due to the invariance of the Haar measure, we can simply absorb the X gates that prepare $|x\rangle = X^{x_1} \otimes \cdots \otimes X^{x_n}|0\rangle =: X(x)|0\rangle$ into the average:

$$Z := \int Z_U \mathrm{d}U = \sum_x \int_U |\langle 0\,|X(x)U|0\rangle|^4 \mathrm{d}U = d \int_U |\langle 0\,|U|0\rangle|^4 \mathrm{d}U . \tag{2.26}$$

To compute the integral, we use second-order Weingarten calculus:

$$\begin{aligned} Z &= d \int_U \mathrm{tr}(|0\rangle\langle 0|^{\otimes 2} U^{\otimes 2} |0\rangle\langle 0|^{\otimes 2} U^{\otimes 2,\dagger}) \mathrm{d}U && (2.27)\\ &= d \sum_{\pi,\sigma \in \mathrm{S}_2} W_{\pi,\sigma}\, \mathrm{tr}(R_\sigma^\dagger |0\rangle\langle 0|^{\otimes 2})\, \mathrm{tr}(R_\pi |0\rangle\langle 0|^{\otimes 2}) && (2.28)\\ &= d \sum_{\pi,\sigma \in \mathrm{S}_2} W_{\pi,\sigma} && (2.29)\\ &= 2d\mathcal{G}_{2,d}^{-1} = 2d \frac{(d-1)!}{(d+1)!} = \frac{2}{d+1} . && (2.30) \end{aligned}$$

Here, we used Lem. 2.3. Note that we haven't actually used the exact form of the Weingarten matrix from Ex. 2.1.2. In fact, along the same lines we find that the average of $\sum_x p(x|U)^k$ is

$$I_k := \sum_x \int p(x|U)^k \mathrm{d}U = k!d\mathcal{G}_{k,d}^{-1} = \frac{k!d!}{(d+k-1)!} . \tag{2.31}$$

Exercise 2.1. *Using Weingarten calculus, compute the operator* $S := d \int (U|0\rangle\langle 0|U^\dagger)^{\otimes 2} \mathrm{d}U$.

2.2 Approximating the Haar measure: Unitary designs

In this chapter, we have seen how Haar integrals over the unitary group can be computed using Weingarten calculus. These tools will help us to design randomized protocols and algorithms for applications, and we will see an example for that already in the next chapter. In practice, Haar-random unitaries are however way too *expensive* to be useful: On a system with n qudits, a quantum

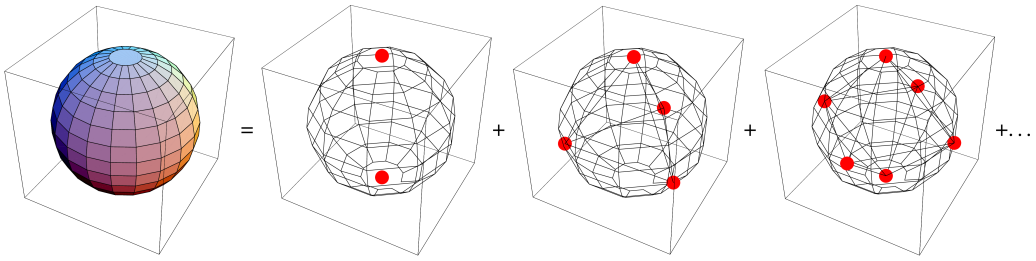


Figure 2.1: Caricature showing a "series expansion" of the Haar measure on the full unitary group (here depicted by a sphere) into finite subsets that agree with the k-th Haar moment. Taken from Kueng and Gross [1].

circuit implementing a Haar-random unitary requires a circuit depth that is exponential in n. This means that any quantum algorithm that utilizes Haar-random unitaries necessarily has exponential runtime.

In this section, we thus briefly discuss a concept introduced to lower the requirements for quantum randomness: *unitary k-designs*. These are sets of unitaries that mimic the Haar measure up to the k-th moment and can thus be used as a replacement in applications that rely on finite moments only. Importantly, unitary designs may be constructed using significantly less resources than Haar-random unitaries, in particular using polynomial-sized circuits.

2.2.1 Unitary designs

Formally, we define a unitary design as follows. Let $\mathcal{G} \subset \mathrm{U}(d)$ be a (finite) set of unitaries (we can extend this to infinite sets equipped with a suitable probability measure). Then, we call $\mathcal{G}$ a *unitary k-design* if

$$\frac{1}{|\mathcal{G}|} \sum_{U \in \mathcal{G}} U^{\otimes k} A U^{\otimes k,\dagger} = \mathsf{M}_k(A)\,, \tag{2.32}$$

for all matrices A. We call the largest k for which Eq. (2.32) holds the *order* of the unitary design. Unitary designs can be seen as a set of points in the unitary group that are "sufficiently equally distributed" to reproduce the first moments of the Haar measure, cf. Fig. 2.1.

Note that the left hand side of Eq. (2.32) can be seen as the k-fold twirl over the set $\mathcal{G}$. This means that averages over the set $\mathcal{G}$ can be computed using averages over the full unitary group – for instance using Weingarten calculus. Vice versa, in applications that only depend on k-fold twirls, Haar-random unitaries can be replaced by unitary k-designs, thereby enabling more resource-efficient and flexible implementations.

One might hope that one could find sufficiently symmetric subgroups of $\mathrm{U}(d)$ that gives natural candidates for unitary designs. However, it turns out that Eq. (2.32) together with the group structure imposes very strict conditions on such a subgroup, resulting in the fact these do no exist for $k \geq 4$ (and $d \geq 5$). Despite this, the most important example of a unitary design (with $k = 3$) is in fact a group, the *Clifford group*, which we will introduce in the next section.

Nevertheless, unitary k-designs exist for all k and dimensions d. Unfortunately, explicit constructions of general unitary k-designs are incredibly rare and the known ones are highly inefficient. To overcome these obstacles, it has been fruitful to demand that Eq. (2.32) holds only approximately – such *approximate unitary designs* can be realized much more easily and enable the efficient implementation of quantum randomness also for higher moments k. We will come back to this point in Ch. 4.

2.2.2 The Clifford group

The prototypical example of a unitary design is the *Clifford group*. Besides designs, the Clifford group plays a major role in quantum error correction, where it typically constitutes the set of "easy" gates in

fault-tolerant quantum computing. Clifford operations can also be efficiently simulated on a classical computer, making them the starting point for classical simulation algorithms and investigations of the "non-classicality" of quantum mechanics.

The simplest way to define the Clifford group is via its local generators: The single-qubit *phase gate* S and *Hadamard gate* H, as well as the two-qubit CNOT gate CX, given by

$$S|x\rangle = i^x|x\rangle \qquad H|x\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^x|1\rangle\big) \qquad CX|x,y\rangle = |x, y\oplus x\rangle\,, \tag{2.33}$$

where $y \oplus x$ denotes addition modulo 2. The group that is generated by S and H on every qubit and CX on every pair of qubits is the n-qubit *Clifford group* Cl_n. It is a finite subgroup of $\mathrm{U}(2^n)$ with $2^{O(n^2)}$ elements. Importantly, every Clifford unitary can be implemented using $O(n^2)$ generators S, H, and CX.

An important subgroup of the Clifford group is the *Pauli group*. Recall the Pauli operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.34}$$

which we complement with the identity $\mathbb{1}$. Then, the multi-qubit Pauli operators on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ are simply given by all possible tensor products of the single-qubit Pauli operators, in formula:

$$\sigma = \sigma_1 \otimes \cdots \otimes \sigma_n\,, \quad \sigma_i \in \{\mathbb{1}, X, Y, Z\}\,. \tag{2.35}$$

The n-qubit Pauli operators form a group up to phases, the so-called n-qubit Pauli group:

$$\mathrm{P}_n := \{i^t \sigma_1 \otimes \cdots \otimes \sigma_n \mid t \in \mathbb{Z}_4,\, \sigma_i \in \{\mathbb{1}, X, Y, Z\}\}\,. \tag{2.36}$$

To see that P_n forms a subgroup of Cl_n, note that $Z = S^2$, $HZH = X$, and $Y = iXZ$. It turns out that we can characterize the Clifford group uniquely through the Pauli group: Clifford unitaries are exactly those unitaries that conjugate Paulis into Paulis, up to a phase.[2] In formula, we thus have

$$\mathrm{Cl}_n.\,\mathrm{U}(1) = \big\{U \in \mathrm{U}(2^n) \mid U\mathrm{P}_n U^\dagger = \mathrm{P}_n\big\}\,. \tag{2.37}$$

Note that we had to add arbitrary global phases ($\mathrm{U}(1)$) to Cl_n as these are present on the right hand side as well.

The qudit case. The definition of the Clifford group can be readily extended to higher-dimensional qudits of dimension q. We will here focus on the case that $q > 2$ is **prime**, as this will matter for the design properties of the Clifford group.

We start by generalizing the generators of the qubit Clifford group. To this end, let $\omega_q := e^{2\pi i/q}$ be a primitive q-th root of unity, and let 2^{-1} be the inverse of 2 modulo q. Define

$$H_q|x\rangle := \frac{1}{\sqrt{q}} \sum_{y=0}^{q-1} \omega_q^{xy}|y\rangle \qquad S_q|x\rangle := \omega_q^{2^{-1}x(x-1)}|x\rangle \qquad CX_q|x,y\rangle := |x, y\oplus x\rangle\,, \tag{2.38}$$

where $y \oplus x$ denotes addition modulo q. Then, we again define the n-qudit Clifford group $\mathrm{Cl}_n(q)$ to be the group generated by H_q, S_q on every qudit and CX_q on every pair of qudits. This is again a finite subgroup of $\mathrm{U}(q^n)$ of order $q^{O(n^2)}$.

Equivalently, we can define $\mathrm{Cl}_n(q)$ in terms of a qudit version of Pauli operators, defined by

$$Z_q|x\rangle := \omega_q^x|x\rangle\,, \qquad X_q|x\rangle := |x\oplus 1\rangle\,, \qquad Y_q := \omega_q^{2^{-1}} X_q^\dagger Z_q^\dagger\,. \tag{2.39}$$

The qudit Pauli group is then given as

$$\mathrm{P}_n(q) := \{\omega_q^{t_0}\sigma_1^{t_1} \otimes \cdots \otimes \sigma_n^{t_n} \mid t_i \in \mathbb{Z}_q,\, \sigma_i \in \{X_q, Y_q, Z_q\}\}\,. \tag{2.40}$$

As in the qubit case, qudit Clifford unitaries map qudit Paulis to qudit Paulis, up to a phase.

[2] In group-theoretic terms, the Pauli group is a normal subgroup of the Clifford group, and the Clifford group is the normalizer of the Pauli group within the unitary group.

The Clifford group as a design. Finally, we have the following result on the design properties of the Clifford group.

Theorem 2.2. *Let q be prime. Then, the Clifford group $\mathrm{Cl}_n(q)$ is a unitary 2-design but not a 3-design if q is odd, and a unitary 3-design but not a 4-design if $q = 2$.*

We will not prove this theorem here. It is not particularly difficult (see e.g. Ref. [2, Sec. 12.2] for a summary), but it requires a slightly deeper analysis of the structure of the Clifford group which shall not be the focus of this course.

2.3 Further reading

To be completed.

CHAPTER 3

MEASURING PROPERTIES OF MANY-BODY STATES

3.1 Shadow estimation: Expectation values through randomized measurements

To be completed.

3.2 Applications

To be completed.

CHAPTER 4

QUANTUM RANDOMNESS II

4.1 Random quantum circuits

To be completed.

CHAPTER 5

RANDOM DYNAMICS IN MANY-BODY SYSTEMS

5.1 Entanglement dynamics and membrane picture

To be completed.

5.2 Quantum scrambling in chaotic systems

To be completed.

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[2] Markus Heinrich. *On stabiliser techniques and their application to simulation and certification of quantum devices*. PhD thesis. Universität zu Köln, 2021.

APPENDIX A

SOME LINEAR ALGEBRA

This section gives a basic introduction to the linear algebraic concepts used in this course. Most of this should already be known from linear algebra and quantum mechanics lectures. At this point, the lectures notes are more detailed than the lecture to achieve a certain self-containement of the notes and provide a reference for later stages of the course.

A.1 States, operators, superoperators

State space As usual, quantum mechanics is modeled on a *Hilbert space* $\mathcal{H}$, which we take, in good quantum info tradition, to be *finite-dimensional* for the remainder of this course. Hence, we can simply think of $\mathcal{H} = \mathbb{C}^d$ with the standard basis $|x\rangle$ labeled by integers $x = 0, 1, \ldots, d-1$, and the standard inner product

$$\langle \psi | \varphi \rangle = \sum_{x=0}^{d-1} \bar{\psi}_x \varphi_x \,, \tag{A.1}$$

where $\psi_x = \langle x | \psi \rangle$ and $\varphi_x = \langle x | \varphi \rangle$ are the coefficients in the standard basis. Typically, we take vectors $\psi \in \mathcal{H}$ to be normalized: $\langle \psi | \psi \rangle = 1$.

The notation of the inner product as a 'bracket' motivates the popular *Dirac* or *bra-ket notation* which we adopt here: In this context, vectors $\psi \in \mathcal{H}$ are called *kets* and written as $|\psi\rangle$. The corresponding *bra* is a dual vector $\langle \psi | \in \mathcal{H}^*$ and given by the linear form $\mathcal{H} \ni \varphi \mapsto \langle \psi | \varphi \rangle$.[1] While the pairing between a bra and ket yields the inner product (the 'bracket'), the pairing between a ket and bra forms a so-called *outer product* $|\psi\rangle\langle\varphi|$ which is a linear operator on $\mathcal{H}$ that acts as $\mathcal{H} \ni \chi \mapsto |\psi\rangle\langle\varphi|\chi\rangle$.

Linear operators The vector space of all linear operators $A : \mathcal{H} \to \mathcal{H}$ is denoted by $\mathrm{L}(\mathcal{H})$. For any $A \in \mathcal{H}$, its *adjoint* $A^\dagger$ is the linear operator for which

$$\langle \psi | A\varphi \rangle = \langle A^\dagger \psi | \varphi \rangle \,, \qquad \forall \psi, \varphi \in \mathcal{H} \,. \tag{A.2}$$

If represented in an orthonormal basis, such as the standard basis, the adjoint operator is the conjugate transpose matrix, $A^\dagger = \bar{A}^\top$.

Definition A.1. *In the following, we define some relevant classes of operators:*

- Hermitian *(or self-adjoint) operator:* $A \in \mathrm{L}(\mathcal{H})$ *such that* $A^\dagger = A$. *Hermitian operators have only real eigenvalues and an orthonormal eigenbasis.*
- Unitary *operator:* $U \in \mathrm{L}(\mathcal{H})$ *such that* $U^\dagger U = UU^\dagger = \mathbb{1}$.
- Positive semi-definite *(psd) operator: Hermitian* $A \in \mathrm{L}(\mathcal{H})$ *with only non-negative eigenvalues. We write* $A \geq 0$.
- Projector*: Hermitian* $P \in \mathrm{L}(\mathcal{H})$ *such that* $P^2 = P$.
- Quantum state*:* $\rho \in \mathrm{L}(\mathcal{H})$ *such that* $\rho \geq$ *and* $\operatorname{tr} \rho = 1$. ρ *is called* pure *if it is a projector:* $\rho^2 = \rho$. *Pure states have rank one and are of the form* $\rho = |\psi\rangle\langle\psi|$.

Finally, the unitaries on $\mathcal{H}$ *form the* unitary group $\mathrm{U}(\mathcal{H}) = \mathrm{U}(d)$.

[1] In mathematics, this is called the *Riesz representation theorem*.

The vector space $\mathrm{End}(\mathcal{H})$ of linear operators on $\mathcal{H}$ forms a Hilbert space of dimension $(\dim \mathcal{H}) = d^2$ in its own right with the *Hilbert-Schmidt* or *trace inner product*:

$$(X|Y) := \mathrm{tr}(X^\dagger Y). \tag{A.3}$$

In particular, we can introduce an orthonormal operator basis as a set of operators $A_1, \ldots, A_{d^2}$ such that $(A_i|A_j) = \delta_{ij}$. We will now introduce an import example of such a basis, the *Pauli basis*.

Example A.1.1: Pauli basis

Recall the Pauli operators

$$\sigma_{0,1} \equiv X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{1,1} \equiv Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_{1,0} \equiv Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.4}$$

which we complement with the identity $\sigma_{0,0} = \mathbb{1}$. Then, the multi-qubit Pauli operators on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ are simply given by all possible tensor products of the single-qubit Pauli operators, in formula:

$$\sigma_a := \sigma_{a_1,a_2} \otimes \cdots \otimes \sigma_{a_{2n-1},a_{2n}}, \quad a \in \mathbb{Z}_2^{2n}. \tag{A.5}$$

Pauli operators are orthogonal, $(\sigma_a|\sigma_b) = 2^n \delta_{a,b}$. In particular, the normalized Pauli operators $\hat{\sigma}_a = 2^{-n/2}\sigma_a$ form an orthonormal operator basis. Note that Pauli operators can be generalized to arbitary dimensions and they give rise to an orthonormal operator basis in any of those.

We leave it as an exercise to show some basic properties of Pauli operators.

Exercise A.1 (Properties of Pauli operators)**.** *Using the definition of Pauli operators, Eq.* (A.5)*, show the following properties:*

(a) $\sigma_a^\dagger = \sigma_a$ *and* $\sigma_a^2 = \mathbb{1}$*, i.e. the multi-qubit Pauli operators are both Hermitian and unitary.*

(b) $\sigma_a \sigma_b \propto \sigma_{a+b}$*, where addition is in* $\mathbb{Z}_2^{2n}$*, i.e. modulo two.*

(c) $\sigma_a \sigma_b = (-1)^{[a,b]} \sigma_b \sigma_a$ *where* $[a,b] := \sum_{i=1}^n a_i b_{n+i} + a_{n+i} b_i$*.*

(d) $(\sigma_a|\sigma_b) = 2^n \delta_{a,b}$*.*

Superoperators and quantum channels Following a common nomenclature, we refer to linear maps $\phi : \mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{H})$ as *superoperators* (on $\mathcal{H}$). We call ϕ *positivity-preserving* or simply *positive* iff $\phi(A) \geq 0$ for all $A \geq 0$. As it turns out, positive maps are not necessarily positive when we let them act on a subsystem of a composite system, i.e. if we consider $\phi \otimes \mathrm{id}_{\mathcal{A}}$ for some auxillary system $\mathcal{A}$. Thus, we say that ϕ is *completely positive* iff $\phi \otimes \mathrm{id}_{\mathcal{A}}$ is positive for any auxillary system $\mathcal{A}$. Completely positive maps are the ones which we consider 'physical', as the map quantum states to quantum states. This leads us to the definition of a quantum channel:

Definition A.2 (Quantum channel)**.** *A quantum channel is a superoperator* ϕ *that is completely positive and trace-preserving, this is* $\phi \otimes \mathrm{id}_{\mathcal{A}}$ *is positive for any auxillary system* $\mathcal{A}$ *and* $\mathrm{tr}(\phi(A)) = \mathrm{tr}(A)$ *for all* $A \in \mathrm{L}(\mathcal{H})$*. We call* ϕ unital *iff* $\phi(\mathbb{1}) = \mathbb{1}$*.*

Example A.1.2: Quantum channels

Some examples of quantum channels are the following:

- *Unitary channels*: $\phi(X) = UXU^\dagger$ for $U \in \mathrm{U}(\mathcal{H})$.
- *Mixed-unitary channels*: $\phi(X) = \sum_i \lambda_i U_i X U_i^\dagger$ for $U_i \in \mathrm{U}(\mathcal{H})$, $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. These are convex combinations of unitary channels.
- *Dephasing channel*: $\phi(X) = \sum_x \langle x|X|x\rangle |x\rangle\langle x|$.
- *Reset channels*: $\phi(X) = \mathrm{tr}(X)\rho$ for a fixed quantum state ρ.

To denote superoperators, it is handy to introduce an 'operator Dirac notation' as follows: In analogy to the usual Dirac notation, we use the Hilbert-Schmidt inner product to define *operator kets and bras* by $|Y) \equiv Y$ and $(X| : Y \mapsto (X|Y)$. Likewise, we can define outer products $|X)(Y|$ which are now linear maps on $\mathrm{L}(\mathcal{H})$, i.e. superoperators, acting as $A \mapsto (Y|A)X$.

The 'operator bra-ket notation' is especially useful to expand a superoperator in an operator basis, i.e. write down its matrix representation. Typically, we will use the (normalized) Pauli basis in this context, but any orthonormal basis works similarly. To this end, we observe that $\mathrm{id} = \sum_a |\hat{\sigma}_a)(\hat{\sigma}_a|$ and thus

$$\phi = \sum_{a,b} |\hat{\sigma}_a)(\hat{\sigma}_a|\phi|\hat{\sigma}_b)(\hat{\sigma}_b| =: \sum_{a,b} \phi_{a,b} |\hat{\sigma}_a)(\hat{\sigma}_b| \tag{A.6}$$

The matrix $(\phi_{a,b})_{a,b}$ is the representation of ϕ in the Pauli basis. For quantum channels, this matrix has certain properties, which we here leave as an exercise:

Exercise A.2. *Let ϕ be a multi-qubit quantum channel and let $(\phi_{a,b})_{a,b}$ be its matrix representation in the Pauli basis. Show that*

(a) $(\phi_{a,b})_{a,b}$ is real.

(b) $\phi_{a,0} = \delta_{a,0}$. If ϕ is unital, it also holds $\phi_{0,b} = \delta_{0,b}$, hence $\phi \simeq \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$.

(c) Suppose ϕ is a Pauli channel*, this is $\phi(X) = \sum_a \lambda_a \sigma_a X \sigma_a^\dagger$ (for convex coefficients λ_a). Then, $(\phi_{a,b})_{a,b}$ is diagonal (use Ex. A.1).*

Norms Throughout this paper, we use *Schatten p-norms* which are defined for any linear map $X \in \mathrm{L}(\mathcal{V}, \mathcal{W})$ between Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$ and $p \in [1, \infty]$ as

$$\|X\|_p := \left(\mathrm{tr}|X|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{d} \sigma_i^p\right)^{\frac{1}{p}}, \tag{A.7}$$

where $|X| := \sqrt{X^\dagger X} \in \mathrm{L}(\mathcal{V})$ and $\sigma_i \geq 0$ are the singular values of X, i.e. the square roots of the eigenvalues of the positive semidefinite operator $X^\dagger X$. In particular, we use the *trace norm* $p = 1$, the *Hilbert-Schmidt norm* $p = 2$, and the *spectral norm* $p = \infty$. The definition of Schatten norms only relies on the Hilbert space structure of the underlying vector space, thus these norms can be defined for operators and superoperators alike.

A.2 Non-orthonormal bases

Let $(f_i)_{i\in[d]}$ be a basis of a Hilbert space $\mathcal{V}$. Thus, every $v \in \mathcal{V}$ has a unique expansion $v = \sum_i v_i f_i$. If $(f_i)_i$ is orthonormal, then the coefficients v_i can be simply expressed as $v_i = \langle f_i|v\rangle$. This can be

generalized to arbitrary bases by introducing the concept of a *dual basis* $(\tilde{f}_i)_i$ which is defined by the linear system of equations

$$\langle \tilde{f}_i | f_j \rangle = \delta_{i,j} \,. \tag{A.8}$$

As $(f_i)_i$ is a basis, this system has a unique solution. It is now straightforward to verify that

$$\langle \tilde{f}_i | v \rangle = \sum_j v_j \langle \tilde{f}_i | f_j \rangle = v_i \,. \tag{A.9}$$

Moreover, this implies that

$$\left(\sum_i |f_i\rangle\langle \tilde{f}_i| \right)(v) = \sum_i v_i f_i = v \,, \tag{A.10}$$

for all $v \in \mathcal{V}$ and hence $\sum_i |f_i\rangle\langle \tilde{f}_i| = \mathrm{id}_{\mathcal{V}}$.

The dual basis can be computed using the *Gram matrix*

$$G_{i,j} := \langle v_i | v_j \rangle \,. \tag{A.11}$$

One can show that G is generally positive semi-definite and since the v_i are linearly independent, the eigenvalues are in fact strictly larger than zero. Hence, it is invertible and we define its inverse as $W := G^{-1}$. Then, the dual basis can be expressed as

$$\tilde{v}_i := \sum_j W_{i,j} v_j \,. \tag{A.12}$$

Indeed:

$$\langle \tilde{v}_i | v_j \rangle = \sum_k W_{i,k} \langle v_k | v_j \rangle = \sum_k W_{i,k} G_{k,j} = \delta_{i,j} \,. \tag{A.13}$$